Homological algebra solutions Week 6

1. Use Baer's Criterion to show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and then give an injective resolution of $\mathbb{Z}$.

   Baer's Criterion: An R-module M is injective if and only if for every ideal $J \subseteq R$, every R-morphism $J \to M$ can be lifted to a morphism $R \to M$.

   Solution: By Baer's Criterion, it suffices to prove the above result for $R = \mathbb{Z}, J = (n)$ and $M = \mathbb{Q}/\mathbb{Z}$. Given $f : n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, one may construct a $\mathbb{Z}$-linear map $\bar{f} : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ by setting $\bar{f}(1) = \dfrac{1}{n} f(n)$. The linearity can be easily verified as $\mathbb{Z}$ is generated by 1 element and it is clear that $\bar{f} = f$ on $n\mathbb{Z}$. As a result, $\mathbb{Q}/\mathbb{Z}$ is injective and one can apply a similar argument to show that $\mathbb{Q}$ is also injective, hence

   $$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

   an injective resolution of $\mathbb{Z}$. $\square$

2. For A an abelian group, we define its *Pontrjagin dual* as : $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. Show that when A is finite, we have $(A^*)^* \cong A$, and deduce that there is an equivalence of categories $\mathbf{FAb} \cong \mathbf{FAb}^{\mathbf{op}}$, where $\mathbf{FAb}$ is the category of finite abelian groups. However, find an abelian group such that $(A^*)^*$ is not isomorphic to A.

   Solution: By classification of finite Abelian groups, we may write A as $A = \bigoplus_{i=1}^{n} \mathbb{Z}_{k_i}$. Since $\mathrm{Hom}_R(-, M)$ commutes with the finite direct sum for R commutative, $A^* = \bigoplus \mathbb{Z}_{k_i}^*$ and it suffices to prove $(\mathbb{Z}_n^*)^* = \mathbb{Z}_n$. First, we show that $\mathbb{Z}_n^*$ can be viewed as a finite subgroup of $\mathbb{Q}/\mathbb{Z}$, identified with H_n defined as follows:

   $$H_n := \{\frac{p}{q} \in \mathbb{Q}/\mathbb{Z} : q|n\}$$

   Through direct computation, one can verify that this is indeed a subgroup with respect to addition. Now consider a linear map $\varphi : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$. φ is uniquely determined by $\varphi(1)$ and $n\varphi(1) = 0$, thus $\varphi(1) \in H_n$ and one can check that each element of H_n defines a such φ. As a result, we have an isomorphism $\varphi \leftrightarrow \varphi(1)$.

Next we show that $H_n^* = \mathbb{Z}_n$. For each $\dfrac{p}{q} \in H_n$, one may define an endomorphism

$$k \cdot (-) : k \cdot (\frac{p}{q}) = \frac{kp}{q} \in H_n \subseteq \mathbb{Q}/\mathbb{Z}$$

which is clearly equal to $k \cdot \mathrm{id}$, the sum of k identities in H_n. While $k \cdot (-) = (k \mod n) \cdot (-)$, we may define a homomorphism $F : \mathbb{Z}_n \to H_n^*$ by sending 1 to id.

To construct the inverse, for $\psi \in H^*$, consider a map $G : \psi \to n\psi(\dfrac{1}{n})$ mod n. Notice that here $\psi(\dfrac{1}{n})$ is assumed to be a rational in $\mathbb{Q}$ in $[0, 1)$ instead of an element in $\mathbb{Q}/\mathbb{Z}$. G is well-defined as each equivalent class of $\mathbb{Q}/\mathbb{Z}$ contains exactly one element in $[0, 1)$ and $G(0) = 0$ trivially. For linearity of G, pick $\alpha, \beta \in H^*$, we have

$$G(\alpha + \beta) = n(\alpha + \beta(1/n)) = n\alpha(1/n) + n\beta(1/n) = G(\alpha) + G(\beta)$$

Hence G is well-defined. One can easily check that $G \circ F = \mathrm{id}_{\mathbb{Z}_n}$ and for $F \circ G$, notice that a map $\psi \in H^*$ is uniquely determined by $\psi(1/n)$ and through direct computation one can see that $\varphi(1/n) = F \circ G(1/n)$. This shows that $\mathbb{Z}_n \cong H^*$.

To show that $(-)^*$ defines an equivalence, notice that $\mathrm{Hom}(-, M)$ defines a contravariant functor, hence $(-)^*$ is a functor from $\mathbf{FAb}^{\mathbf{op}}$ onto $\mathbf{FAb}$. However, it also defines a functor from $\mathbf{FAb}$ to $\mathbf{FAb}^{\mathbf{op}}$: for a map $f : A \to B$, it is sent to $\mathrm{Hom}(f, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. By inverting the arrows, we get a map $\mathrm{Hom}(f, \mathbb{Q}/\mathbb{Z})^{\mathbf{op}} : \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z})$ in the opposite category. Thus $(-)^*$ defines an equivalence between $\mathbf{FAb}$ and $\mathbf{FAb}^{\mathbf{op}}$ and $(-)^*$ is equivalent to $(-)^{*\mathbf{op}}$ as $(A^{*\mathbf{op}})^* \cong A$, which also implies $\mathrm{Hom}(A, B) \cong \mathrm{Hom}((A^{*\mathbf{op}})^*, (B^{*\mathbf{op}})^*)$ on the category of abelian groups. The isomorphism between $((-)^*)^{*\mathbf{op}}$ and $\mathrm{id}_{\mathbf{FAb}^{\mathbf{op}}}$ can be proven analogously hence we are finished.

To give a counterexample of non-finite abelian groups such that $(A^*)^* \neq A$, we may consider $\mathbb{Z}$. It is clear that $\mathbb{Z}^* = \mathbb{Q}/\mathbb{Z}$, however $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \neq \mathbb{Z}$, since otherwise $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is generated by id as a $\mathbb{Z}$-module, yet there exists a $\mathbb{Z}$-linear map that sends $\frac{p}{q}$ to $\frac{p}{nq}$, for n an integer. This morphism is not generated by the identity since all maps generated by the identity takes the form $p/q \to n \cdot p/q$ $\square$

3. Let $F : \mathbf{A} \to \mathbf{B}$ be a right exact functor and$U : \mathbf{B} \to \mathbf{C}$ be an exact functor.If $\mathbf{A}$ has enough projectives, show that we have a natural isomorphism :

$$L_i(UF) \cong U(L_i(F))$$

Solution: Let $X \in \mathbf{A}$ be an object and $P_\bullet \xrightarrow{f_\bullet} X$ be a projective resolution, then we first show that

$$U(L_i(FX)) = U(\ker Ff_i/\text{im } Ff_{i+1}) = \ker UFf_i/\text{im } UFf_{i+1} = L_i(UFX)$$

It suffices to show that $U(\ker Ff_i) \cong \ker UFf_i$ and $U(\text{im } Ff_{i+1}) \cong \text{im } UFf_{i+1}$, since then we have $U(\ker Ff_i/\text{im } Ff_{i+1}) \cong \ker UFf_i/\text{im } UFf_{i+1}$. As the proof is similar we will only prove the case of kernels. Let $f : Y \to Z$ be a morphism in $\mathbf{B}$. Consider the following rows of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & U\ker f \arrow[r] \arrow[d, "g"] & UY \arrow[r] \arrow[d, equal] & UZ \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & \ker Uf \arrow[r] & UY \arrow[r] & UZ
\end{tikzcd}$$

The first row is exact by exactness of U and the second row is exact by definition of kernel. The map g is induced by universal property of kernels and one can verify that the diagram is commutative. By 5-lemma we have $\ker Uf = U\ker f$. Now we may assume that $f = f_i$ and replace Y, Z by FP_{i+1}, FP_i. This finishes the proof.
To show that there is a natural transformation between the two functors,given $P_\bullet \xrightarrow{a_\bullet} X, Q_\bullet \xrightarrow{b_\bullet} Y$ and a map $f : X \to Y$(notice that this induces a morphism between the projective resolutions),one needs to show that the following diagram is commutative:

$$\begin{tikzcd}
U(L_iFX) \arrow[r, "U(L_iFf)"] \arrow[d, "\tau_X"] & U(L_iFY) \arrow[d, "\tau_Y"] \\
L_i(UFX) \arrow[r, "L_i(UFf)"] & L_i(UFY)
\end{tikzcd}$$

Where τ_X, τ_Y are isomorphisms obtained previously.To do this one may consider the following diagram:

$$\begin{tikzcd}
 & U(L_nFX) \arrow[dr, "\tau_X"] \arrow[ddd, dashed, "UL_n(F\varphi)"] & \\
U\ker Ff_n \cong \ker UFf_n \arrow[ur, "q"] \arrow[rr, "i"] \arrow[d, "UF\varphi_n"] & & L_n(UFX) \arrow[d, "L_n(UF\varphi)"] \\
U\ker Fg_n \cong \ker UFg_n \arrow[rr, "j"] \arrow[dr, "p"] & & L_n(UFY) \\
 & U(L_nFY) \arrow[ur, "\tau_Y"] &
\end{tikzcd}$$

The map $UL_n(F\varphi)$ can be viewed as the natural map induced by $UF\varphi_n$ between the kernels and all sequares,triangles are commutative except the pair $(L_n(UF\varphi) \circ \tau_X, \tau_Y \circ U(L_nF\varphi))$. But since q(respectively q) is the natural quotient morphism, it is an epimorphism and to verify that $L_n(UF\varphi)\circ\tau_X = \tau_Y \circ U(L_nF\varphi)$, it suffices to verify that $L_n(UF\varphi)\circ\tau_X\circ q = \tau_Y \circ U(L_nF\varphi)\circ q$. This can be done through diagram chasing:

$$\begin{aligned} L_n(UF\varphi) \circ \tau_X \circ q &= L_n(UF\varphi) \circ i \\ &= j \circ UF\varphi_n \\ &= \tau_Y \circ p \circ UF\varphi_n \\ &= \tau_Y \circ U(L_nF\varphi) \circ q \end{aligned}$$

This finishes the proof. $\square$

4. Let R be a commutative ring, and N an R-module. Show or recall that $-\otimes_R N : R-\mathbf{Mod} \to R-\mathbf{Mod}$ is right exact. We denote by $\mathrm{Tor}_R^i(-, N)$ the associated left derived functor. Find a projective resolution of $\mathbb{Z}/n\mathbb{Z}$, and use it to compute $\mathrm{Tor}_{\mathbb{Z}}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for every $i \geq 0$ and $m \in \mathbb{Z}$.

   Solution: we show that $-\otimes N$ is right exact. Consider an exact sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{\pi} M'' \to 0$$

   pass this sequence by the functor $-\otimes N$, one gets

$$0 \to M' \otimes N \xrightarrow{i\otimes N} M \otimes N \xrightarrow{\pi\otimes N} M'' \otimes N \to 0$$

   To prove the exactness, one may use the following two lemmas:
   Lemma 1: For M, N, P arbitrary R-modules, one has $\mathrm{Hom}(M\otimes N, P) = \mathrm{Hom}(M, \mathrm{Hom}(N, P))$.
   Proof: One may construct a pair of isomorphisms between the two Hom modules as follows: for $\varphi : M \otimes N \to P$, there is an induced map $F(\varphi) : M \to \mathrm{Hom}(N, P)$,such that $F\varphi(m)(n) = \varphi(m \otimes n)$. Conversely, given a map $\psi : M \to \mathrm{Hom}(N, P)$, there is an induced map $G(\psi) : M \otimes N, P$ such that $G(\psi)(m \otimes n) = \psi(m)(n)$.One can easily check that the two are well-defined R-homomorphisms that are inverse to each other.$\square$

   Lemma 2: A sequence $M' \xrightarrow{f} M \xrightarrow{\pi} M'' \to 0$ is exact if and only if for all R-module N the following sequence is exact:

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{\bar{\pi}} \mathrm{Hom}(M, N) \xrightarrow{\bar{f}} \mathrm{Hom}(M', N)$$

   Proof: $\Rightarrow$: The proof is trivial by definition of right exactness.
   $\Leftarrow$: *Surjectivity of* π: Suppose that π is not surjective then let $N = M''/\mathrm{im}\ \pi$, and let q be the projection onto N, then clearly $0 \circ \pi = q \circ \pi$, hence $\bar{\pi}$ is not injective, giving us a contradiction;
   $\ker \bar{f} = \mathrm{im}\ \bar{\pi}$: We have $\mathrm{im}\ \bar{\pi} \subseteq \ker \bar{f}$ and it suffices to check that $\ker \pi \subseteq$

im $\bar{f}$. To do this, first let $N = M/\text{im } f$ and let φ be the natural projection, then $\varphi \circ f = 0 \implies \varphi \in \ker \bar{f} = \text{im } \bar{\pi}$. Hence there is a $\psi : M'' \to M/\text{im } f$, such that $\psi \circ \pi = \varphi$. As a result, one has $\ker \pi \subseteq \ker \psi = \text{im } f$. $\square$

By Lemma 2, to show the right exactness of tensor product, it suffices to show the exactness of the following sequence for every R-module P:

$$\text{Hom}(M'' \otimes N, P) \to \text{Hom}(M \otimes N, P) \to \text{Hom}(M' \otimes N, P) \to 0$$

By Lemma 1, this sequence is equivalent to

$$\text{Hom}(M'', \text{Hom}(N, P)) \to \text{Hom}(M, \text{Hom})(N, P) \to \text{Hom(M',Hom(N,P))} \to 0$$

Yet the last sequence is naturally exact since $\text{Hom}(-, \text{Hom}(N, P))$ is a left exact functor.

For the following part the tensor product $- \otimes -$ is always tensor product of $\mathbb{Z}$-modules unless specified. Consider the following projective resolution of $\mathbb{Z}_n$:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

Pass it by $- \otimes \mathbb{Z}_m$, one gets

$$0 \to \mathbb{Z}_m \xrightarrow{n} \mathbb{Z}_m \to \mathbb{Z}_n \otimes \mathbb{Z}_m \to 0$$

Suppose that m, n are not coprime. Using $M/IM = M \otimes R/I$, one can show that $\mathbb{Z}_n \otimes \mathbb{Z}_m = \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}/\text{gcd}(m, n)\mathbb{Z}$. As a result, for $i \geq 2$, $\text{Tor}_i(\mathbb{Z}_n, \mathbb{Z}_m) = 0$; By right exactness we have $\text{Tor}_0(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\text{gcd}(n,m)}$. Moreover $\text{Tor}_1(\mathbb{Z}_n, \mathbb{Z}_m) = \ker(\cdot n) \otimes \mathbb{Z}_m$ More explicitly, we may write this kernel as $\{x \in \mathbb{Z}_m : nx = 0 \mod m\} = (\text{scm}(m, n)/n)\mathbb{Z}_m = (m/\gcd(m, n))\mathbb{Z}_m \cong \mathbb{Z}/(\text{gcd}(m, n))\mathbb{Z}$.

For m, n coprime, since $\text{gcd}(m, n) = 1$, we have $m\mathbb{Z}_n = \mathbb{Z}_n$, hence $\mathbb{Z}_n \otimes \mathbb{Z}_m = 0$ and the multiplication by n is invertible on $\mathbb{Z}_m$. Thus $\cdot n$ is an isomorphism and $\text{Tor}^i(\mathbb{Z}_n, \mathbb{Z}_m) = 0, \forall i \geq 0$ $\square$